LEXICOGRAPHIC SHELLABILITY FOR BALANCED COMPLEXES

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ABSTRACT. We introduce a notion of lexicographic shellability for pure, balanced boolean cell complexes, modelled after the CL-shellability criterion of Björner and Wachs for posets [BW2] and its generalization by Kozlov [Ko2] called CC-shellability. We give a lexicographic shelling for the quotient of the order complex of a Boolean algebra of rank $2n$ by the action of the wreath product $S_2 \wr S_n$ of symmetric groups, and we provide a partitioning for the quotient complex $\Delta(\Pi_n)/S_n$.

Stanley asked for a description of the symmetric group representation β_S on the homology of the rank-selected partition lattice Π_n^S in [St2], and in particular he asked when the multiplicity $b_S(n)$ of the trivial representation in β_S is 0. One consequence of the partitioning for $\Delta(\Pi_n)/S_n$ is a (fairly complicated) combinatorial interpretation for $b_S(n)$; another is a simple proof of Hanlon's result [Ha] that $b_{1,\dots,i}(n) = 0$. Using a result of Garsia and Stanton from [GS], we deduce from our shelling for $\Delta(B_{2n})/S_2 \wr S_n$ that the ring of invariants $k[x_1, \dots, x_{2n}]^{S_2 \wr S_n}$ is Cohen-Macaulay over any field k.

1. INTRODUCTION

Let B_n denote the Boolean algebra of subsets of $\{1, \cdots, n\}$ ordered by inclusion and let Π_n be the lattice of unordered partitions of $\{1, \cdots, n\}$ ordered by refinement. The natural symmetric group action on $\{1, \cdots, n\}$ induces an action on each of these posets. Likewise, the wreath product $S_2 \wr S_n$ acts on the elements of the Boolean algebra B_{2n}. Any rank-preserving group action on a finite, ranked poset P with minimal and maximal elements $\hat{0}$ and $\hat{1}$ induces an action on the order complex $\Delta(P)$, that is, on the simplicial complex consisting of an $(i-1)$-face for each i-chain $\hat{0} < u_1 < \cdots < u_i < \hat{1}$ in P; the group action on poset elements induces an action on chains. This gives rise to a quotient cell complex, denoted $\Delta(P)/G$, which is comprised of the G-orbits of order complex faces.

Note that $\Delta(P)/G$ need not coincide with the order complex of the quotient poset P/G because there may be covering relations $u < v$ and $u' < v'$ in P belonging to distinct orbits despite having $u' = gu$ and $v' = g'v$ for some $g, g' \in G$. Babson and Kozlov give conditions under which $\Delta(P)/G = \Delta(P/G)$ in [BK]. Equality does not hold for $P = \Pi_n, G = S_n$ and for $P = B_{kn}, G = S_k \wr S_n$, so the quotient complexes $\Delta(\Pi_n)/S_n$ and $\Delta(B_{2n})/S_2 \wr S_n$ that we will consider are not simplicial complexes and in particular are not order complexes of posets.

We shall give a lexicographic shelling for the quotient complex $\Delta(B_{2n})/S_2 \wr S_n$ and a partitioning for $\Delta(\Pi_n)/S_n$, using a generalized notion of chain-labelling for balanced complexes. By way of comparison, Ziegler showed in [Zi] that the quotient poset $\Delta(\Pi_n/S_n)$ is not Cohen-Macaulay for $n \geq 19$. We will verify in Section 3 that

$\Delta(B_6)/S_3 \wr S_2$ is not shellable. It was shown in [GS], [Re], [St2] that shellings and partitionings for quotient complexes $\Delta(P)/G$ yield information about sub-rings of invariant polynomials and about G-representations on the homology of $\Delta(P)$. Our approach is to extend poset lexicographic shelling methods to a general enough class of complexes to include quotient complexes $\Delta(P)/G$, taking advantage of properties quotient complexes have in common with posets.

To this end, we introduce a notion of CC-labelling for pure, balanced boolean cell complexes in Section 2 and confirm that it induces a lexicographic shelling. We require three conditions for a chain-labelling to be a CC-labelling. The first of these is a direct translation of the poset requirement that each interval must have a unique (topologically) increasing chain and that this be lexicographically smallest on the interval. The second condition, which we call the "crossing condition", is automatic for posets, and we verify that it also holds for all quotient complexes. The third condition, the "multiple-face-overlap" condition, is vacuous for simplicial complexes. In the quotient complex $\Delta(B_{2n})/S_2 \wr S_n$ this technical condition follows readily from the increasing chain condition because of the nature of the ascents that occur in the labelling for $\Delta(B_{2n})/S_2 \wr S_n$. A virtually identical argument isolates exactly where our lexicographic order on facets in $\Delta(\Pi_n)/S_n$ fails to satisfy the multiple-face-overlap condition. It also helps us find a face whose link is the real projective plane, implying $\Delta(\Pi_n)/S_n$ is not Cohen-Macaulay over $\mathbb{Z}/2\mathbb{Z}$, and hence is not shellable.

Section 3 provides a CC-labelling for $\Delta(B_{2n})/S_2 \wr S_n$ while Section 4 gives a chain-labelling for $\Delta(\Pi_n)/S_n$ that satisfies the increasing chain condition. Section 4 constructs from this labelling a partitioning for $\Delta(\Pi_n)/S_n$. In Section 6, we express the multiplicity $b_S(n)$ of the trivial representation in the S_n-representation on the homology of the rank-selected complex $\Delta(\Pi_n^S)$ in terms of the flag h-vector for the quotient complex $\Delta(\Pi_n)/S_n$. This gives a combinatorial interpretation for $b_S(n)$, since $h_S(\Delta(\Pi_n)/S_n)$ is the number of facets contributing minimal new faces of support S. In any lexicographic shelling (or a partitioning which uses descents in a similar spirit), this support S is the set of topological descents in each facet. One may use this to show that $b_S(n) > 0$ for a particular S by showing that the descent set S arises in some lexicographic shelling (or partitioning) step (cf. [HH] for more results of this nature). In Section 7, an analysis of which S arise as minimal new faces in a partitioning yields a simple proof of Hanlon's result that $b_{1,\ldots,i}(n) = 0$. Finally, Section 8 gives an application to subrings of invariant polynomials by applying a result of Garsia and Stanton in [GS].

The remainder of our introduction will review the notion of boolean cell complex, lexicographic shellability for posets and related terminology. Boolean cell complexes were introduced by Björner in [Bj] and by Garsia and Stanton in [GS]. Stanley studied their face posets, namely simplicial posets, in [St3]. He defined the face ring (also called Stanley-Reisner ring) of a boolean cell complex and then showed that a boolean cell complex is Cohen-Macaulay if and only if its face ring is Cohen-Macaulay. Duval studied free resolutions of face rings of boolean cell complexes in [Du]. Reiner developed a theory of P-partitions for Coxeter groups in order to shell (or in some cases partition) quotients of Coxeter complexes in [Re]. Our interest is in lexicographic shelling for balanced, boolean cell complexes.

Definition 1.1 (Björner, Garsia-Stanton). *A regular cell complex is* **boolean** *if every interval in its face poset is a boolean algebra.*

A boolean cell complex is much like a simplicial complex, except that more than one face may have the same set of vertices. We note that boolean cell complexes may alternatively be defined quite naturally in terms of simplicial sets. We find it convenient to refer to i-cells as i-faces, 0-cells as vertices, and in the same vein to call cells of top dimension facets. A boolean cell complex is **pure** of dimension n if all the maximal cells have dimension n, and then it is **balanced** if the vertices may be colored with $n + 1$ colors so that no two vertices in a face have the same color. We refer to the set of colors for the vertices in a face as the **support** of the face and say a face has **disconnected** support if the support includes i, k for $i < k$ and does not include some j for $i < j < k$.

Björner established the following notion of shellability (phrased slightly differently) for boolean cell complexes in [Bj1].

Definition 1.2 (Björner). *A boolean cell complex is **shellable** if the facets may be ordered $F_1, \ldots, F_k$ so that $F_j \cap (\cup_{i=1}^{j-1} F_i)$ is pure of codimension one for each $1 < j \leq k$.*

Recall that the order complex of a finite, ranked poset is a balanced simplicial complex in which vertices are colored by poset rank. This will allow us to translate poset notions of lexicographic shellability to conditions on the order complex which may then be extended to give shelling criteria for more general balanced complexes.

One may conclude from the existence of a shelling that a boolean complex has the homotopy type of a wedge of spheres of top dimension and is Cohen-Macaulay (cf. [Bj1]). The arguments are similar (though slightly more subtle) to those for simplicial complexes.

Proposition 1.1 (Björner). *If a pure boolean cell complex is shellable, then it has the homotopy type of a wedge of spheres of top dimension.*

In a shelling, each facet either attaches along its entire boundary, "closing off" a sphere, or its overlap with the union of earlier facets is a simplicial complex with a cone point, implying the homotopy type is unchanged by the facet's insertion.

Proposition 1.2 (Björner). *If a pure boolean cell complex is shellable, then it is Cohen-Macaulay.*

This may be shown by shelling its barycentric subdivision (cf. [Bj2, p. 173]), which is a simplicial complex, then invoking Munkres' classical result [Mu2, p.117,121-123] that Cohen-Macaulayness does not depend on choice of triangulation.

Definition 1.3 (Björner). *An integer labelling of the covering relations in a finite poset with $\hat{0}$ and $\hat{1}$ is an **EL-labelling** if it has the following two properties, which together constitute the **increasing chain condition**.*

 (1) *Every interval has a unique saturated chain with (weakly) increasing edge labels.*
 (2) *The increasing chain is the lexicographically smallest chain of labels on an interval.*

A **chain-labelling** is a labelling of poset covering relations such that the label assigned to a covering relation $u \prec v$ may depend on the choice of root, namely on the saturated chain $\hat{0} \prec u_1 \prec \cdots \prec u_k = u$ as well as on u and v. Recall from [BW1], [BW2] that a **CL-labelling** is any chain-labelling satisfying the increasing-chain condition. An edge-labelling or chain-labelling induces a partial order on

facets by lexicographically ordering the sequences of labels assigned to saturated chains. It is shown in [Bj1] (resp. [BW2]) that any total order extension of the lexicographic order given by an EL-labelling (resp. CL-labelling) is a shelling order on facets.

Kozlov generalized poset EL-shellability (resp. CL-shellability) in [Ko2] to a criterion he called **EC-shellability** (resp. **CC-shellability**) by relaxing the requirement that every poset interval must have a unique increasing chain. In effect, he instead requires each interval to have a unique saturated chain that behaves topologically like an increasing chain with respect to the chosen lexicographic order. We rediscovered EC/CC-shellability in the course of joint work with Kleinberg (see [HK]); it will be convenient for the fairly involved shelling arguments in later sections to use the notation and point of view taken in [HK], so now let us review terminology from [HK].

We classify ascents and descents in a poset chain-labelling (or edge-labelling) λ as follows: let us say that a pair of edges $u \prec v$ and $v \prec w$ constitute a **topological ascent** if the word consisting of two consecutive labels $\lambda(u, v)$ and $\lambda(v, w)$ is lexicographically smallest on the interval from u to w and let us say that the pair of covering relations $u \prec v$ and $v \prec w$ comprise a **topological descent** otherwise. We may further distinguish between topological ascents with increasing or decreasing consecutive labels by calling the former **honest ascents** and the latter **swap descents**. Similarly, we call topological descents with decreasing labels **honest descents** and all others **swap ascents**. In this language, a poset is EC-shellable (resp. CC-shellable) if each interval (resp. rooted interval) has a unique topologically increasing chain (namely a chain consisting entirely of topological ascents) and this is the lexicographically smallest chain on the interval. It is shown in [Ko2] that these labellings induce lexicographic shellings, for the same reason that EL-labellings and CL-labellings do.

2. A LEXICOGRAPHIC SHELLING CONDITION FOR BALANCED BOOLEAN CELL COMPLEXES

In this section, we extend CC-shellability to pure, balanced boolean cell complexes and EL/CL/EC/CC-shellability to pure, balanced simplicial complexes. We always choose indices so that F_i precedes F_j lexicographically for $i < j$. When the vertices $v_1, \ldots, v_t$ in a face σ are colored $c_1, \ldots, c_t$, then we call the set $S = \{c_1, \ldots, c_t\}$ the **support** of σ. It will be convenient to represent an arbitrary color set as $s_0, \ldots, r_1, s_1, \ldots, r_2, s_2, \ldots, r_k, s_k, \ldots, r_{k+1}$ for some $s_0 \geq 1$, $r_{k+1} < n$ and with $s_i - r_i > 1$ for all $1 \leq i \leq k$. We use $s_i, \ldots, r_{i+1}$ to denote the collection of all possible colors from s_i to r_{i+1}. Thus, the colors not in S are those between r_i and s_i for some i, and the colors that are smaller than s_0 or larger than r_{k+1}.

Let us begin by translating the poset lexicographic shellability condition of Björner and Wachs [BW1] to a condition on the order complex so as to make an analogous condition for pure, balanced boolean cell complexes. Consider any finite, graded poset with unique minimal and maximal elements $\hat{0}$ and $\hat{1}$. Notice that the increasing chain condition on the Hasse diagram of a poset may be viewed as a condition on the order complex. We make the following conventions:

(1) The label assigned to a poset covering relation from rank i to $i+1$ is placed on the consequent order complex edge colored $i, i+1$ for $1 \leq i \leq n-1$.

(2) The label of each poset edge involving $\hat{0}$ (resp. $\hat{1}$) is assigned to the corresponding vertex colored 1 (resp. $n-1$).

(3) The interval from u to v in the poset is the collection of faces colored $rk(u), rk(u)$
$+ 1, \ldots, rk(v) - 1, rk(v)$ which include the vertices u and v.

(4) Let $i = rk(u)$ and $j = rk(v)$. Then each poset saturated chain from u to v translates to a walk on the resulting face colored $i, i+1, \ldots, j-1, j$ in the order complex. Such a walk along edges colored $i', i'+1$ for $1 \leq i' < j$ passes through the vertex colors sequentially.

The increasing-chain condition on an interval from rank i to j amounts to a condition on all the faces in the order complex consisting of vertices colored $i, \ldots, j$ which include a particular pair of vertices colored i and j. This requirement makes sense for arbitrary pure, balanced simplicial complexes, using the balancing to play the role of poset rank. Any pure, balanced simplicial complex will be lexicographically shellable if it satisfies this increasing chain condition along with another requirement which we call the crossing condition. For balanced boolean cell complexes, we must define the notion of interval a little bit more carefully, but again the increasing chain condition will be a similar requirement on cells in an interval; the increasing chain condition together with the crossing condition and a third requirement called the multiple-face-overlap condition will imply that a pure, balanced boolean cell complex is lexicographically shellable.

Let us generalize the notions of interval and rooted interval to balanced complexes, as follows.

Definition 2.1. *Let τ be a face colored $1, \ldots, i, j$ for some $i < j$ in a pure, balanced boolean cell complex. The **rooted interval** specified by τ is the collection of faces colored $1, \ldots, j$ that contain τ.*

Notice that a cell complex which is not a simplicial complex might have several faces comprised of the same vertex set of support $1, \ldots, i, j$. Each of these faces gives rise to a different interval. For this reason, it does not seem wise to allow edge-labellings and un-rooted intervals when workin with boolean cell complexes that are not simplicial complexes.

Definition 2.2. *Let u, v be a pair of vertices colored i, j for $i < j$ in a pure, balanced simplicial complex. Then (unrooted) **interval** specified by u and v is the collection of faces colored $i, \ldots, j$ which contain the vertices u and v.*

Next, we adapt the definition of topological ascent and descent to balanced boolean cell complexes.

Definition 2.3. *A facet F_j has a **topological descent** at the color r if there is a codimension one face in $F_j \cap (\cup_{i<j} F_i)$ omitting only the vertex colored r. Otherwise, it has a **topological ascent** at rank r.*

In a poset, one may view the saturated chains as non-self-intersecting paths from $\hat{0}$ to $\hat{1}$ in the Hasse diagram, so then saturated chains intersect where two of these paths cross each other. Notice that when two poset saturated chains cross $c-1$ times in the proper part of the poset, one obtains 2^c distinct saturated chains by choosing which of the two saturated chains to follow on each of the c segments between consecutive crossing points. The existence of these poset chains implies for lexicographic orders that every maximal face in $F_j \cap (\cup_{i<j} F_i)$ skips a single

interval of consecutive ranks. The crossing condition is designed to test for this behavior in arbitrary balanced complexes.

Notice in the case of boolean cell complexes that are not simplicial complexes that the crossing condition, given next, does not always ensure that maximal faces in $F_j \cap (\cup_{i<j} F_i)$ have support skipping a single interval of consecutive colors. Specifically, it does not apply to faces $\sigma \in F_j \cap F_i$ of support $1, \ldots, r_1, s_1, \ldots, r_2, \ldots, s_k, \ldots, r_{k+1}$ such that another face in $F_j \cap F_i$ has support $1, \ldots, r+1$. The multiple-face-overlap condition accounts for these faces.

Definition 2.4. *A balanced boolean cell complex is* **CC-shellable** *if there is a chain-labelling satisfying the following three conditions.*

(1) **Increasing chain condition.** *Each rooted edge-interval has a unique extension with topologically increasing labels and this is the lexicographically smallest face in the interval (i.e. its lexicographically earliest extension to a facet is lexicographically smallest among facets that may be obtained from faces in the interval).*

(2) **Crossing condition.** *Let σ be a face in the intersection of a facet F_k with a lexicographically earlier facet F_j. Suppose that (1) the support of σ includes $1, \ldots, r$ while no other face in $F_j \cap F_k$ includes support $1, \ldots, r'$ for $r' > r$ and (2) the complement of σ has disjoint support. Then there is some facet F_i for $i < k$ and some face $\tau \in F_i \cap F_k$ such that $\sigma \subsetneq \tau$ and the complement of τ has connected support $r+1, \ldots, s$ for some $s \geq r+1$.*

(3) **Multiple-face-overlap condition.** *Suppose the intersection of two facets F_i and F_j (with $i < j$) contains two faces σ, τ, such that σ is maximal in $F_j \cap (\cup_{i'=1}^{j-1} F_{i'})$ and τ is maximal in $F_j \cap F_i$. Furthermore, assume that σ has support including the colors $1, \ldots, r'$ for some $1, \ldots, r'$ which is not a subset of the support of τ. Then τ must be contained in a codimension one face γ of F_j such that $\gamma \in F_j \cap (\cup_{i'=1}^{j-1} F_i')$. Letting $1, \ldots, r, s, \ldots, n$ denote the support of τ, it suffices to check this for $s \leq r'$.*

The final remark in the multiple-face-overlap condition will be invaluable to our proofs in later sections and is confirmed within the proof of Theorem 2.1. It allows us to assume when some $F_i \cap F_j$ includes maximal faces σ and τ as above that the first covering relation of F_j skipped in τ has larger label than the covering relation of the same rank in F_i (for $i < j$). To prove such a face σ has codimension one, we may assume (to get a contradiction) that the interval of F_j skipped by σ consists entirely of topological ascents.

Remark 2.1. The above criterion specializes to pure, balanced simplicial complexes, in which case the multiple-face-overlap condition is vacuously true. For balanced simplicial complexes, the above criterion is easily modified to give notions of EL/CL/EC-shellability as follows. For EL/CL-shellability, we require increasing chains instead of merely topologically increasing chains. For EL/EC-shellability, we label edges in a way that does not depend on the root, and intervals are specified by pairs of vertices rather than also depending on the entire root.

We call any ordering of the facets of a balanced complex which is induced by an EL/CL/EC/CC-labelling a **lexicographic shelling**. Let us check that these do indeed give shellings.

Theorem 2.1. *If $F_1, \ldots, F_r$ is a lexicographic shelling for a pure, balanced boolean cell complex Δ of dimension $n-1$, then $F_l \cap \left(\cup_{k=1}^{l-1} F_k\right)$ is pure of codimension one for each l, so $F_1, \ldots, F_r$ is a shelling.*

PROOF. Let H be a maximal face in $F_j \cap (\cup_{i<j} F_i)$, so $H \subseteq F_j \cap F_{i'}$ for some $i' < j$. It suffices to show that H has codimension one in F_j. Let $S = \{s_0, \ldots, r_1, s_1, \ldots, r_{k+1}\}$ be the support of H. The crossing condition implies that the complement of S is a single interval of consecutive ranks, except in the following scenario. The crossing condition does not apply if (1) the support of H includes $1, \ldots, r$ but excludes $r+1$, and (2) there is some other face $H' \subseteq F_j \cap F_{i'}$ has support including $1, \ldots, r+1$. However, in this case the multiple-face-overlap condition ensures that H has codimension one in F_j, as desired. Hence, we only need to consider H of support $S = \{1, \ldots, r, s, \ldots, n\}$ or $S = \{s_0, \ldots, n\}$ for some $s_0 > 1$ or $S = \{1, \ldots, r_1\}$ for some $r_1 < n$. Let us assume H has support $1, \ldots, r, s, \ldots, n$, since the other arguments are similar. The facet $F_{i'}$ must be strictly smaller in lexicographic order than F_j on the interval skipped by H, since $F_j \cap F_{i'}$ does not include any faces of support $1, \ldots, r+1$. Thus, the increasing chain condition ensures that F_j must have a topological descent on the rooted interval specified by H restricted to color set $1, \ldots, r, s$. Let $F_{i''}$ be the facet in which one such topological descent is replaced by a topological ascent. $F_{i''}$ precedes F_j in lexicographic order, and $F_j \cap F_{i''}$ contains a face of codimension one in F_j which contains H. Hence, H must be codimension one in F_j to be maximal in $F_j \cap (\cup_{i<j} F_i)$.

Next, we verify the remark in the multiple-face-overlap condition, recalling that $\tau \subseteq F_i \cap F_j$ for some F_i preceding F_j in lexicographic order. Since τ is assumed to have support $1, \ldots, r, s, \ldots, n$, the increasing chain condition implies that either τ has a topological descent on the interval from color r to s, or τ is lexicographically smallest on this interval. In the former case, we argue as above. In the latter case, $F_i \cap F_j$ must agree up to color s, namely there must be a face $\sigma \subseteq F_i \cap F_j$ whose support includes $1, \ldots, s$, just as asserted. □

Remark 2.2. Just as in a poset lexicographic shelling, the topologically decreasing chains give rise to facets attaching along their entire boundaries, and the homotopy type of a lexicographically shellable balanced boolean cell complex is a wedge of spheres of top dimension, indexed by the topologically decreasing chains.

One could define CL-shellability for balanced complexes by requiring all of the topological ascents (resp. topological descents) in a CC-shelling to be actual ascents (resp. descents). This, however, would give a more restrictive notion of CL-shellability than the one very recently introduced by Hultman in [Hu].

We conclude this section by confirming that quotient complexes always satisfy the crossing condition. Later sections will give a lexicographic shelling for the quotient complex $\Delta(B_{2n})/S_2 \wr S_n$ and use lexicographic shelling ideas in a partitioning for $\Delta(\Pi_n)/S_n$. Lexicographic shellability for another class of balanced boolean cell complexes, the nerves of ranked, loop-free small categories, is discussed in [HK].

Proposition 2.1. *Quotient complexes $\Delta(P)/G$ satisfy the crossing condition.*

PROOF. Suppose that two saturated chain orbits F_j, F_k (with $j < k$) share a maximal face σ of support $S = \{s_0, \ldots, r, s_1, \ldots, r_2, s_2, \ldots, r_{k+1}\}$. We may assume $s_0 = 1$, by assumption (1) of the crossing condition. Then there must be poset saturated chains C_j, C_k belonging to the orbits F_j, F_k, respectively, such that C_j, C_k

also share a face of support S. Now consider the saturated chain C_i in P which agrees with C_j on ranks $1, \ldots, s_1$ and agrees with C_k on ranks $s_1, \ldots, n$. Denote the orbit of C_i by F_i. We may assume that F_j and F_k do not share a face colored $\{1, \ldots, r+1\}$, by assumption (1) of the crossing condition. Thus, F_j already has a strictly earlier labelling than F_k on the interval up to the color s_1. Since F_i agrees with F_j on color set $1, \ldots, s_1$, F_i also must precede F_k lexicographically. Restricting F_i to color set $1, \ldots, r, s_1, \ldots, n$ gives the desired face τ.   □

3. A LEXICOGRAPHIC SHELLING FOR $\Delta(B_{2n})/S_2 \wr S_n$

An edge-labelling for the lattice B_{2n} of subsets of $\{1, \ldots, 2n\}$ ordered by inclusion comes from labelling each covering relation $\{\sigma_1, \ldots, \sigma_{i-1}\} \subseteq \{\sigma_1, \ldots, \sigma_{i-1}, \sigma_i\}$ with the number $\sigma_i \in \{1, \ldots, 2n\}$ being inserted. This labelling assigns the permutation $\sigma_1 \sigma_2 \cdots \sigma_{2n} \in S_{2n}$ to the saturated chain $\emptyset \prec \{\sigma_1\} \prec \cdots \prec \{\sigma_1, \ldots, \sigma_{2n}\}$. If the numbers $1, \cdots, 2n$ are placed in a table, as in Figure 1, then each element of $S_2 \wr S_n$ may be viewed as the composition of some $\pi_1 \in S_{2n}$ permuting the elements of each row with a permutation π_2 permuting the rows. Thus, $\pi_1 =$

1	2
3	4
5	6

FIGURE 1. Labelled boxes acted upon by $S_2 \wr S_3$

$(12)^{e_1}(34)^{e_2} \cdots (2n-1, 2n)^{e_n}$ for some choice of $e_i \in \{0, 1\}$ for each $1 \le i \le n$, and there is some $\pi \in S_n$, such that $\pi_2(2i) = 2\pi(i)$ and $\pi_2(2i-1) = 2\pi(i) - 1$ for $1 \le i \le n$.

The action of $S_2 \wr S_n$ on $\{1, \ldots, 2n\}$ induces an action on the saturated chains in B_{2n}. Let us denote orbit representatives of this action by the permutations in S_{2n}, written in one-line notation, which label the chosen saturated chains in B_{2n}. We choose the saturated chain labelled by the lexicographically smallest possible permutation in an orbit as the orbit representative. The permutations in S_{2n} occurring as labels are the ones with the property that $2i - 1$ comes before both $2i$ and $2i + 1$ for $1 \le i < n - 1$ and that $2n - 1$ comes before $2n$. The labelling for orbits is a chain-labelling using the labels assigned to the orbit representative.

Example 3.1. The orbit representatives for $\Delta(B_6)/S_2 \wr S_3$, listed in lexicographic order, are 123456, $1235 \bullet 46$, $123 \circ 56 \bullet 4$, $13 \bullet 2456$, $13 \bullet 25 \bullet 46$, $13 \bullet 256 \bullet 4$, $1 \circ 34 \bullet 256$, $1 \circ 345 \bullet 26$, $1 \circ 3456 \bullet 2$, $135 \bullet 246$, $13 \circ 5 \bullet 26 \bullet 4$, $1 \circ 35 \bullet 4 \bullet 26$, $1 \circ 35 \bullet 46 \bullet 2$, $13 \circ 56 \bullet 24$, and $1 \circ 3 \circ 56 \bullet 4 \bullet 2$. Hollow dots denote swap ascents while filled-in dots indicate descent locations. For instance, the swap ascent in $1 \circ 3456 \bullet 2$ comes from a codimension one face skipping rank 1 in the intersection of 134562 with 132564, resulting from the fact that 312564 is in the same orbit as 134562.

In the shelling argument for $\Delta(B_{2n})/S_2 \wr S_n$, we refer to the ith row as being **empty** at the element u in a saturated chain $\hat{0} \prec \{\sigma_1\} \prec \cdots \prec \{\sigma_1 \cdots \sigma_k\} = u$ if $2i - 1, 2i \notin \{\sigma_1, \cdots, \sigma_k\}$. Similarly we call row i **half-full** at u if $2i - 1 \in \{\sigma_1, \cdots, \sigma_k\}$ but $2i \notin \{\sigma_1, \cdots, \sigma_k\}$, and we refer to row i as **full** at u if $2i - 1, 2i \in \{\sigma_1, \cdots, \sigma_k\}$.

All the saturated chain orbits belonging to the same rooted edge-interval $\hat{0} \prec u_1 \prec \cdots \prec u_k = u < v$ must agree in the following three ways:

(1) The same collection of half-full rows of u must be full in v
(2) The same number of rows must switch from empty in u to full in v.
(3) The same number of rows must switch from empty in u to half-full in v.

The first of these three conditions depends on our use of edge-intervals which are rooted, since any two half-full rows in u are equivalent, but they are distinguishable within the context of a saturated chain orbit from $\hat{0}$ to u . In order to verify the increasing-chain condition below, we will show that there cannot be two (topologically) increasing chains that agree in all three ways.

Theorem 3.1. *The labelling of saturated chains with minimal orbit representatives gives a lexicographic shelling. More specifically, it is a CC-shelling.*

PROOF. The crossing condition follows from Proposition 2.1. Next we classify topological ascents and descents in order to verify the increasing chain condition.

We claim that every descent is an honest descent. Replacing a descent $\sigma_i \sigma_{i+1}$ by the ascent $\sigma_{i+1} \sigma_i$ yields a lexicographically smaller permutation, thus a member of a different orbit (since the orbit representative was already the lexicographically smallest member of its orbit). In fact, the permutation obtained by this swap is the orbit representative of a new orbit since no new ascents have been introduced by swapping σ_i and σ_{i+1}, which means that the requirement that $2j - 1$ come before $2j$ and $2j + 1$ for all j is still satisfied.

Whenever an orbit representative F_j has $2i - 1$ immediately preceding $2i + 1$ and then later has $2i + 2$ before $2i$, there is a swap ascent at the node between $2i - 1$ and $2i + 1$. This is because the facet F_i with $2i + 2$ and $2i$ swapped is lexicographically earlier, and the facets F_i, F_j share a codimension one face which omits the node between $2i - 1$ and $2i + 1$. Notice that the characterization of this node as a swap ascent depends on later ranks in the chain. This is not a problem, since the labels themselves, and thus the lexicographic order on facets, only depends on the root of the chain. It is not hard to check that all other ascents are honest ascents, and we have already shown that there are no swap descents.

Now we must verify that each rooted interval has a unique (topologically) increasing chain. If two orbit representatives both had only topological ascents on a rooted interval from u to v, each would have labels in increasing order and be free of swap ascents of the type described above. To avoid descents, each increasing chain must begin by completing all the requisite half-filled rows of u in increasing order, and then begin new rows, proceeding sequentially. The lexicographically smallest chain in the rooted edge-interval proceeds through all the empty rows of u to be filled in v before turning to empty rows of u to be half-filled in v. Any lexicographically later orbit on the interval which is free of descents would also first complete the set of half-full rows of u to be filled in v and then would proceed to new rows. To differ from the lexicographically smallest orbit it would at some point necessarily insert one element into a new row immediately before inserting two elements to the next new row. This would yield a swap ascent, as discussed above. For example, the orbit labelled 134 comes after the increasing chain labelled 123 in the interval $(\emptyset, \{1, 2, 3\})$ in $B_4/S_2 \wr S_2$, but 134 has a swap ascent at 13 since 4 comes before 2 in any orbit including 134. Thus, the lexicographically smallest orbit on the interval is the only topologically increasing orbit, just as desired.

Finally, we must confirm the multiple-face-overlap condition. Suppose two faces σ and τ are maximal in some $F_i \cap F_j$ and that σ is maximal in $F_j \cap (\cup_{i' < j} F_{i'})$. We may assume that τ has support $1, \ldots, r, s, \ldots, 2n - 1$, because we will show (in the last paragraph of our proof) that for $\Delta(B_{2n})/S_2 \wr S_n$ every maximal face in $F_j \cap (\cup_{i' < j} F_{i'})$ has support omitting a single interval. Assume also that σ has support including $1, \ldots, r'$ for some $r' \geq s$ such that no face in $F_i \cap F_j$ has support $1, \ldots, r'+1$. In particular, this means that F_j agrees with F_i on the interval skipped by τ. Let u be the element of rank r' in F_j, let v be the element of rank $r' + 1$ in F_j and let v' be the element of rank $r' + 1$ in F_i. Notice that $u \prec v$ is equivalent to $u \prec v'$ in τ, but not in σ. This is only possible if two rows are equivalent in τ but not in σ, i.e. if two different rows switch from empty to half-full on the interval skipped by τ. Furthermore, the earlier of these rows (call it ρ') must be completed in the step $u \prec v'$ while the later row (denoted by ρ) is completed in the step $u \prec v$. We may assume that the interval skipped by τ has no topological descents, for otherwise τ would be contained in a codimension one face of $F_j \cap (\cup_{i' < j} F_{i'})$, and we would be done.

Thus, all the letters inserted into rows to be half-filled in the interval of F_j skipped by τ are inserted by progressing through these rows sequentially. If ρ' is half-filled immediately before ρ, then there is a codimension one face within F_j that contains τ and belongs to $F_i \cap F_j$. This face is obtained by half-filling ρ' and ρ in a single step. Otherwise, let $\rho' = \rho_1, \rho_2, \ldots, \rho_k = \rho$ be the sequence of consecutive rows that are half filled between ρ' and ρ in F_j. There must be some ρ_{m-1}, ρ_m for $1 < m \leq k$ such that ρ_m is complete before ρ_{m-1} in F_j, since ρ is completed before ρ' in F_j. There also must be an $F_{i'}$ which agrees with F_j except that it reverses the order in which the second elements of rows ρ_{m-1} and ρ_m are inserted. This $F_{i'}$ comes before F_j lexicographically. There is a codimension one face of F_j that contains τ and is contained in $F_j \cap F_{i'}$. It is obtained by half-filling ρ_{m-1} and ρ_m in a single step. Thus, τ is contained in a codimension one face belonging to $F_j \cap (\cup_{i' < j} F_{i'})$, as desired.

There are no facets F_j such that $F_j \cap (\cup_{i < j} F_i)$ has a maximal face whose complement has disjoint support because this would mean that skipping disjoint intervals allows two covering relations to be identified that could not be identified by skipping a single interval. Such identification could only come from making two rows interchangeable, but because the rows have length two, it suffices to skip the single interval beginning where the first row is half-filled and ending where the second row is half-filled. Thus, we have checked all the necessary conditions for a lexicographic shelling.   □

The story is quite a bit different for $\Delta(B_{kn})/S_k \wr S_n$ when k is greater than 2. The increasing chain condition fails for the lexicographic order on orbit representatives chosen to be lexicographically as small as possible. Consider the first four facets $F_1 = 123456, F_2 = 124356, F_3 = 124536, F_4 = 124563$ in $\Delta(B_6)/S_3 \wr S_2$. The intersection $F_4 \cap (F_1 \cup F_2 \cup F_3)$ has two maximal faces σ, τ where σ is colored $1, 2, 3, 4$ and τ is colored $4, 5$, so τ has codimension greater than 1. Here, $\sigma \subseteq F_4 \cap F_3$ and $\tau \subseteq F_4 \cap F_3$.

Indeed, $\Delta(B_6)/S_3 \wr S_2$ cannot possibly be shellable. Molien's Theorem (cf. [St1]) together with code given to us by Vic Reiner allowed us to determine the Hilbert series for the ring of invariants $k[\Delta(B_6)]^{S_3 \wr S_2}$ with generators graded by poset rank; we obtained $(1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + q^8)(1/((1 - q)(1 - q^2)(1 - q^3)(1 - q^4)(1 -$

$q^5)(1 - q^6)))$. However, if $\Delta(B_6)/S_3 \wr S_2$ had a shelling, then Theorem 6.2 of [GS] (stated in Section 8) would also yield this Hilbert series as follows. The product $1/((1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5))$ accounts for polynomials in $\theta_1, \theta_2, \theta_3, \theta_4$ and θ_5, as discussed in Section 8 while the numerator $1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + q^8$ accounts for elements of the basic set given by a theorem of [GS], described in Section 8. That is, each shelling step F_i with minimal new face G_i of support S_i contributes q^{d_i} to the numerator where $d_i = \sum_{x \in S_i} x$.

In the case of $\Delta(B_6)/S_3 \wr S_2$, there is no ordering on facets that would give rise to the necessary collection of exponents $0, 2, 3, 4, 4, 5, 6, 6, 7, 8$, so there cannot be a shelling. To see this, let us represent facets by lexicographically smallest orbit representatives. Note that whichever facet comes last among $142536, 142563, 145236$ and 145263 will contribute a minimal new face whose support includes ranks $1,3$ and 5. This implies that the last step among these four would contribute an exponent of at least 9 to the numerator of the Hilbert series, a contradiction since q^8 is the largest power of q present.

4. A LEXICOGRAPHIC ORDER ON FACETS OF $\Delta(\Pi_n)/S_n$

First we briefly describe which chains in Π_n are in the same orbit and then give a way of representing chain orbits by trees. After this, we give a chain-labelling on $\Delta(\Pi_n)/S_n$ in terms of this tree-representation for the facets of $\Delta(\Pi_n)/S_n$. We remark that a similar tree-representation appears in [Ko3].

4.1. **Orbits of partition lattice chains.** The S_n-orbit of a partition in Π_n is determined by the size of its blocks, i.e. by an integer partition. Thus, vertices in $\Delta(\Pi_n)/S_n$ are given by unordered partitions of the integer n. However, two edges $u < v$ and $u' < v'$ in $\Delta(\Pi_n)$ may belong to distinct S_n-orbits even if $u' \in Orb(u)$ and $v' \in Orb(v)$, because this does not guarantee the existence of a single permutation $\pi \in S_n$ such that $\pi(u) = u'$ and $\pi(v) = v'$. If no such π exists, then $u < v$ and $u' < v'$ give rise to distinct edges in $\Delta(\Pi_n)/S_n$ which have the same vertices. Thus, faces in $\Delta(\Pi_n)/S_n$ are sequences of successively refined partitions of the integer n enriched with some additional information.

Example 4.1. Consider a chain $22 \prec 11|11 \prec 3|8|11$ with numbers denoting block sizes. The orbit does not depend on which block of size 11 is split into blocks of size $3, 8$. However, there are two orbits of the form $22 \prec 11|11 \prec 3|8|11 \prec 3|8|3|8 \prec 3|4|4|3|8$, because the two blocks of size 8 are not interchangeable since they were created at different ranks. On the other hand, the two blocks of size 8 are indistinguishable within the chain which skips immediately from $11|11$ to $3|8|3|8$.

The orbit of a chain keeps track of what type of block is split into what types of pieces at each rank in a chain. To be precise, the **type** of a block in u_k will be its S_n-equivalence class, relative to a chain $\hat{0} < u_1 < \cdots < u_k < \hat{1}$. Two blocks b_1, b_2 in u_k are said to be S_n-**equivalent** if (1) they have the same content, (2) they must be created in the same refinement step $u_i < u_{i+1}$, and (3) they must be children of blocks B_1 and B_2, respectively, which are themselves equivalent to each other. This includes the possibility that $B_1 = B_2$, so b_1, b_2 come from a single parent. The point is that S_n may swap the elements of b_1 with the elements of b_2 in a way that preserves the chain. Denote the orbit of a chain C by $\pi(C)$.

4.2. **Two encodings for faces in $\Delta(\Pi_n)/S_n$.** First we encode the orbit of a chain as a tree whose nodes are the partition blocks that occur in the chain. The root is the single block appearing as $\hat{0}$ in any poset chain. The children of a tree node B are the blocks obtained from B when it is refined in the chain. Each tree node is labelled by its block content, and each parent is also labelled with the rank at which it is refined. The equivalence class of a particular block relative to a chain orbit comes from the tree built by that chain orbit. Two blocks are equivalent if there is a graph automorphism that swaps the blocks and carries each tree block to one with identical labels. The orbit of a saturated chain gives rise to a binary tree.

The labelling given next will depend on a choice of planar embedding for these trees. We will define this embedding by specifying for each parent an ordering on its children. Let us make a convention for choosing a planar embedding so as to assign a label $\lambda(\pi(\hat{0} \prec \cdots \prec u \prec v))$ to each covering relation orbit $\pi(u \prec v)$ based on the entire saturated chain orbit $\pi(\hat{0} \prec \cdots \prec u)$ to which $\pi(u \prec v)$ belongs. In the process of choosing the label, we shall also choose a planar embedding for the tree given by $\pi(\hat{0} \prec \cdots \prec u \prec v)$. In particular, this embedding will give us an ordering on the blocks for each partition in the chain, which depends only on the root of that chain. The chain-labelling, provided next, uses this ordering on the blocks of a partition to be refined in a covering relation.

Sometimes, we will use a more compact encoding for a chain. Namely, we list n balls in a row with $n-1$ bars separating them, and place numbers between 1 and $n-1$ below the bars. The balls represent the n numbers being partitioned, since these are freely interchangeable. The numbers below the bars record ranks at which bars are inserted while progressively refining a partition. This representation for a chain often is not unique, but it completely determines the chain orbit. We will order the blocks in the fashion described at the beginning of Section 4.3, so this will determine our choice of representation. See Figure 4 for an example.

4.3. **A chain-labelling which nearly satisfies the increasing chain condition.** Let us assign labels to orbits $\pi(\hat{0} \prec u_1 \prec \cdots \prec u_k = u \prec v)$ of rooted covering relations. The distinct $\pi(u \prec v)$ with a fixed root $\pi(\hat{0} \prec u_1 \prec \cdots \prec u_k = u)$ are specified by which type of block from u is split (recalling that type means equivalence class) together with the content of its children. Assume by induction that a saturated chain orbit $\pi(\hat{0} \prec u_1 \prec \cdots \prec u_k = u)$ imposes an order on the blocks of u. We obtain from this an ordering on the blocks of v for each $u \prec v$ as follows:

(1) Split the leftmost block B in the ordered partition of u which belongs to the equivalence class to be split by the covering relation $u \prec v$.
(2) Place the lexicographically smaller of the two blocks obtained from B to the left of an inserted bar separating the two blocks resulting from B.

Thus, we get an ordering on the blocks of v from the ordering for u by replacing B by the two blocks derived from it ordered lexicographically and otherwise preserving block order. One may then use the position where the new bar is inserted to give a chain-labelling.

Remark 4.1. This labelling by bar position is motivated by a feature of Π_n (which is related to the splitting basis for the partition lattice given in [Wa]). Each permutation $\pi \in S_n$ gives rise to a boolean sublattice B_{n-1} of those partitions obtained by listing $1, \ldots, n$ in the order given by π (written in one-line notation). Each

partition consistent with π is specified by choosing a subset of the $n-1$ possible bars to insert splitting the numbers into blocks.

The labelling by bar position does not always satisfy the increasing chain condition, as indicated by Example 4.2. To transform this labelling into one which will satisfy the increasing chain condition, we will introduce a block-sorting step next.

Example 4.2. Consider $\Delta(\Pi_{22})/S_{22}$. Take the rooted interval $\hat{0} \prec u < v$ with $u = 11|11$ and $v = 1|1|9|2|2|7$ where one block of size 11 splits into $1|1|9$ and the other into $2|2|7$. This rooted interval includes the product of chains in Figure 2 with chain-labels as shown. The left half gives labels for saturated chains beginning with $11|11 \prec 1|10|11$ and the right half for those beginning with $11|11 \prec 2|9|11$. The lexicographically smallest chain appears farthest to the left and is labelled $1, 2, 13, 15$. The rightmost chain in Figure 2 consists entirely of honest ascents, even in light of all the saturated chains in the interval rather than only those depicted in Figure 2. Thus, this chain-labelling violates the increasing chain condition on the interval from $11|11$ to $1|1|9|2|2|7$. Figure 2 shows the chains in the interval
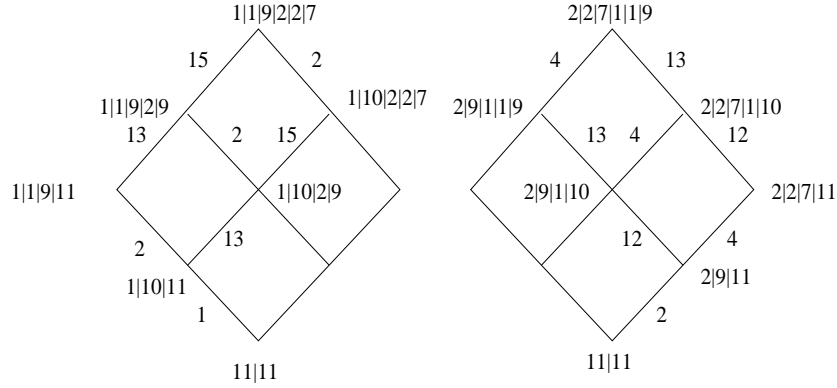

FIGURE 2. First approximation at a chain-labelling

in which bars are inserted left to right. Covering relations are labelled by bar positions. However, the labelling does not give a shelling because the intersection of the rightmost chain with those coming earlier is not pure of codimension one.

4.4. **A modified chain-labelling which satisfies the increasing chain condition.** We will add a block sorting step just before bar position is recorded. This will yield the labelling in Figure 3 for the product of chains from Figure 2. When we split a block of size 11 into blocks of size 2,9, and then break the other block of size 11 into blocks 1,10, we sort the blocks of size 1,10 to the left of the blocks of size 2,9 before assigning the second label.

Our chain-labelling assumes (by induction) that we have already assigned a label to the orbit $\pi(\hat{0} \prec u_1 \prec \cdots \prec u_k = u)$ and in the process have ordered the blocks of u. It then specifies a label for $u \prec v$ as well as an ordering on the blocks of v.

  (1) Block refinement: Split the leftmost block B belonging to the equivalence class to be split in $u \prec v$ into two blocks, the smaller of which is placed on the left. Thus, a block B is replaced by two blocks derived from it with
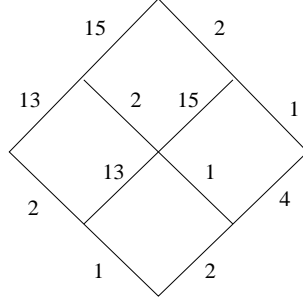
FIGURE 3. Revised chain-labelling which incorporates block sort

the smaller on the left, giving a more refined ordered partition. Note that
block equivalence in orbits of saturated chains only comes from a single
block splitting into two identical pieces.

(2) Block sort: If B is equivalent to another block B' in u_i for some $i < k$,
let $P(B)$ be the common parent of B and B'. Compare the subtree of
descendents of B to the subtree of descendents of B' to decide whether B
or B' should be the left child of $P(B)$ in the block order for v. The left
child is the block with the lexicographically smaller word comprised of the
positions at which bars occur within that block. Ties are broken using the
rank at which B and B' were first split (the block with earlier rank is sorted
farther to the left). Next, apply this sorting procedure to the successive
ancestors of B.

(3) Chain-labelling: The label assigned to $\pi(\hat{0} \prec \cdots \prec u \prec v)$ is a 3-tuple with
the following components, listed in order of precedence:
  (a) The post-sort position of the newly inserted bar.
  (b) The word consisting of all the post-sort bar positions in v.
  (c) The (ordered) list of ranks at which the successive ancestors of B were
  themselves split, with $P(B)$ given lowest precedence.

By Proposition 2.1, $\Delta(\Pi_n)/S_n$ satisfies the crossing condition, so let us turn
our attention to the increasing chain condition and multiple-face-overlap condition.
The proofs of Theorem 4.1 and Theorem 4.2 adopt several ideas from [HK].

Theorem 4.1. *The above labelling satifies the increasing chain condition.*

PROOF. Denote the above labelling by λ. Notice that any topologically increasing
chain on a rooted interval $\hat{0} \prec \cdots \prec u < v$ must insert bars from left to right
(post-sort) into u in such a way that each block of u is split into pieces which are
nondecreasing in size from left to right. There is at least one such increasing chain,
obtained by greedily inserting bars from left to right so that the children of each
block are nondecreasing in size from left to right. Whenever two equivalent blocks
are to be refined, refine them in such a way that the second one to be refined does
not get sorted to the left of the first one refined. That is, refine the first of these
two blocks in the way that gives a smaller left child as soon as the two refinements
differ. What remains to show is that there is a unique topologically increasing chain
on each interval and that this chain is lexicographically earliest on the interval. To
this end, we will show that any saturated chain that is not lexicographically first

on an interval has a topological descent. Our argument will repeatedly use the fact that sorting never moves newly inserted bars to the right.

Suppose that the orbit $\pi(C)$ of a saturated chain $\hat{0} \prec u_1 \prec \cdots \prec u_k \prec \cdots \prec w$ is not lexicographically smallest on a rooted edge-interval $\hat{0} \prec u_1 \prec \cdots \prec u_k < w$. Consider $u \in \pi(C)$ of lowest rank such that the rooted edge $\pi(\hat{0} \prec \cdots \prec u \prec v)$ in $\pi(C)$ has larger chain-label than $\pi(\hat{0} \prec \cdots \prec u \prec v')$ for some $v' \neq v$ in a different saturated chain orbit also belonging to the rooted edge-interval $\pi(\hat{0} \prec \cdots \prec u_k < w)$. Choose v' with the minimal possible label among all such choices. Let B be the block from u that is split in the step $u \prec v$ and let B' be the block (also from u) that is split in $u \prec v'$. We consider three cases, depending on whether the labels $\lambda(u \prec v), \lambda(u \prec v')$ first differ in the first, second or third component of the label. We will sometimes abuse notation by not listing the root, even though the labellings do depend on them.

Case I: Suppose the labels differ in the (post-sort) bar position. We further subdivide this case, depending on whether the blocks B, B' being split by $u \prec v$ and $u \prec v'$, respectively, are equal or not. Case Ia: If $B = B'$, then we will show that $\pi(C)$ must either have an honest descent or a swap ascent within the interval. Let b' be the smaller of the two blocks resulting from splitting B' in $u \prec v'$; note that b' must eventually be derived from B or an equivalent block later in $\pi(C)$, within the interval (since $u \prec v'$ belongs to the same interval). Before this happens, there will be a descent or swap ascent since this later step will have a lexicographically smaller label than any step before it in the interval. This is because the block B (or an equivalent block from which b' is obtained) is at this point sorted at least as far to the left as it would be when the bar is inserted in $u \prec v'$. Thus, the bar splitting off b' must be to the left of the bar insertion immediately before it (and thus there must be a descent) unless there is another step splitting B into two larger pieces earlier in $\pi(C)$ such that b' comes from the right component in this split block. In this case, there must either be a swap ascent in B immediately before b' is split off on its own, or there must be a descent if there are steps splitting B, then splitting another block, then later splitting a block derived from B to create b'.

Case Ib: If $B \neq B'$, then eventually we will split B'. At this point, either we break off the smallest piece of B', just as in v', yielding the smallest label so far in $\pi(C)$ restricted to the interval and thus a descent, or else the preceding $B = B'$ argument may be applied to the remainder of $\pi(C)$ within the interval to obtain a topological descent at a higher rank.

Case II: Now suppose $u \prec v'$ has the same post-sort bar position as $u \prec v$, but that their labels differ in the words made up of post-sort bar positions in v and v'. Then $B \neq B'$ when we let B, B' be the blocks which are split in $u \prec v$, $u \prec v'$, respectively. Eventually, a bar must be inserted in $\pi(C)$ into a block which is equivalent to B' to create a smallest possible piece b', as above. Note that equivalent blocks in a saturated chain orbit must be adjacent regardless of how they are split and sorted, by virtue of the underlying binary tree of blocks. Therefore, when b' is derived from B' later in $\pi(C)$, sorting will still move the new bar to the position it would have achieved in v', unless B' is split into two larger pieces and then b' is derived later from the larger of these. When we are not in this special case, then the bar position word when b' is created will be smaller than that for $u \prec v$, and the bar to the right of b' is placed at least as far to the left as it would be in $u \prec v'$. Thus, $\pi(C)$ must have a descent on the interval. In the special case

where b' is derived from a descendent of B' which is not the leftmost descendent, there still will be a swap ascent or a descent. The former must occur if one of the steps splitting B' occurs immediately before b' is split off. The latter must occur if B' is split, then some B not descending from B' is split and then later B' is further refined.

Case III: Now suppose that the first two coordinates agree, but that the earliest distinct ancestor of B' is split at an earlier rank than that of B. This means B and B' are not equivalent. Hence, blocks of both types must be split in any saturated chain orbit in the interval, yielding a descent or swap ascent in $\pi(C)$ eventually, since the block B' is sorted to the left of B when it is split. If B' is split immediately after $u \prec v$, then there is a lexicographically smaller chain with these two steps reversed, i.e. a topological descent. Otherwise, any intermediate steps will have larger labels, and the chain must have a descent (or swap ascent) by similar reasoning to above. Thus, we have verified the increasing chain condition in all cases. $\square$

Corollary 4.1. *The quotient complex $\Delta(\Pi_n)/S_n$ has lexicographic shelling steps at all facet insertions except those which violate the multiple-face overlap condition.*

Next we describe precisely where the multiple-face-overlap condition fails.

Theorem 4.2. *The above labelling also satisfies the multiple-face-overlap condition, except in the case of certain facets with identical blocks created in consecutive steps from a single parent.*

PROOF. We will show that the multiple-face-overlap condition holds assuming that the two blocks to be identified have the same parent block. Suppose $F_j \cap F_k$ has two maximal faces σ, τ such that σ is maximal in $F_k \cap (\cup_{i<k} F_i)$ and τ has support including $1, \ldots, r'$ where no face in $F_j \cap F_k$ has support $1, \ldots, r'+1$. We need only check the condition if σ has support $1, \ldots, r, s, \ldots, n$ for some $s < r'$. We may also assume F_k consists entirely of topological ascents on the interval from rank r to s. Otherwise, σ has codimension one, and we would be done. Thus, the post-sort positions of the bars inserted within F_k from rank r to s are increasing left to right. Let u be the element of rank r in F_k. Then the bars inserted into any particular block of u between rank r and s are also arranged so that blocks are nondecreasing in size from left to right.

Note that the splitting step between ranks r' and $r'+1$ in F_j must not be equivalent to the splitting step in F_k. However, when we restrict to σ, these steps become equivalent relative to the chain σ. Thus, the blocks being split must have the same content. Let us call these blocks B_j and B_k, respectively. If B_j and B_k are created in consecutive steps in the interval of F_k from rank r to s, then we get a codimension one face belonging to $F_j \cap F_k$ which contains σ. This face is obtained by skipping the step immediately after B_j is created and immediately before B_k, so that these are created in a single step from the same block. If B_j and B_k are not created consecutively in F_k but do have the same parent, then there is some block B_{k-1} created immediately before B_k in F_k which is also equivalent to B_k in σ. Consider the facet $F_{j'}$ with $j' < k$ which agrees with F_k except that the roles of B_k and B_{k-1} are reversed from rank r' onward. $F_{j'}$ comes before F_k lexicographically because the covering relations where they first differ share the same first two coordinates of their label. Because B_{k-1} was created earlier, the

third component of the label is smaller for $F_{j'}$. Since B_{k-1} and B_k are created in consecutive steps in F_k in the interval skipped by σ, the codimension one face creating B_{k-1} and B_k in a single step in F_k will be in the intersection $F_k \cap (\cup_{k' < k} F_{k'})$ and will contain σ, as desired.   □

Section 4 will discuss what to do when B_j and B_k come from distinct parent blocks. It will use a partitioning for real projective space and some generalizatinos to incorporate non-shelling steps into the partitioning. First, we give an example of a face in $\Delta(\Pi_8)/S_8$ whose link is $\mathbb{R}P_2$, implying for $n \geq 8$ that $\Delta(\Pi_n)/S_n$ is not Cohen-Macaulay over $\mathbb{Z}/2\mathbb{Z}$, and hence is not shellable. The above analysis helped us find this example.

Example 4.3. Consider the chain

$$12345678 < 1234|5678 < 1|234|5|678 < 1|2|34|5|6|78 < 1|2|3|4|5|6|7|8$$

in Π_8. Its S_8-orbit is a face in $\Delta(\Pi_8)/S_8$ whose link is the real projective plane. To see this, notice that the link has three vertices: one at rank 1, one at rank 3 and one at rank 5, and it has 6 edges and 4 2-simplices, all arranged as $\mathbb{R}P_2$. This is essentially the same construction used in Proposition 3.1 of [He] to study $\Delta(B_{lm})/S_l \wr S_m$, so more detail may be found there.

Notice that our labelling never gives any chains consisting entirely of topological descents. This is just as one would expect, given that $b_{1,\dots,n-2}(n)$ vanishes [Ha], [St2] and that $\Delta(\Pi_n)/S_n$ is collapsible [Ko1].

5. Partitioning $\Delta(\Pi_n)/S_n$

Theorem 4.1 showed that the only way a non-shelling step F_j may arise is when the multiple-face-overlap condition is violated. More specifically, Theorem 4.2 showed that the facet F_j must at some rank refine a block which in the context of a face $F \subseteq F_j$ is equivalent to a block to its left. For example, consider the face of support $2, 4$ within the facet shown in Figure 4.

Definition 5.1. *A* **partitioning** *of a pure boolean cell complex Δ is an assignment to each facet F_i of one of its faces G_i, so that the boolean upper intervals $[G_i, F_i]$ give a partitioning of the faces in the complex, i.e., so that Δ may be written as a disjoint union $[G_1, F_1] \cup \cdots \cup [G_s, F_s]$.*

A partitioning of a pure, balanced boolean cell complex Δ gives a combinatorial interpretation for each flag h-vector coordinate $h_S(\Delta)$ as the number of facets F_i whose minimal face G_i has support S. In this section, we give a partitioning for $\Delta(\Pi_n)/S_n$ as follows. We show how to partition subcomplexes of links of faces coming from progressively more general classes of non-shelling steps. Then we use the topological ascents and descents of the lexicographic order of Section 4 to extend and merge these link subcomplex partitionings into a partitioning for $\Delta(\Pi_n)/S_n$.

FIGURE 4. Facet involving two equivalent blocks

En route to a partitioning for $\Delta(\Pi_n)/S_n$, Section 5.1 will give partitionings for subcomplexes of certain links. For example, consider the link of the face τ of nsupport $2, 4, 6, 8, 10, 11, 12, 13, 14, 15$ contained in the facet depicted in Figure 4. This facet will contribute to our partitioning an interval $[G_j, F_j]$ with G_j of support $2, 5, 9$.

5.1. **Partitioning subcomplexes of links arising in non-shelling steps.** Consider the face F comprised of the following elements. First it has a partition into m blocks of size $k + 1$ along with a single block of size l for some $l > k + 1$. Then, for each $1 \le j \le k$, the chain includes a partition in which the block of size l is unrefined and each of the other m blocks is split into j singletons and a single block of size $k + 1 - j$. Finally, the chain includes a partition whose only nontrivial nontrivial block is the block of size l, and a saturated chain upward from this rank which sequentially splits off singletons from the lone nontrivial block. Let Δ denote the subcomplex of the link of F consisting of those chains which initially insert from left to right the bars separating the m equivalent blocks.
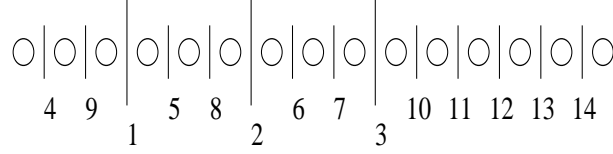

FIGURE 5. Facet with slot permutations $123, 123, 321$

The facets in Δ correspond to $(k + 1)$-tuples $(\sigma_0, \ldots, \sigma_k)$ of permutations in S_m with the identity as the first permutation. Requiring σ_0 to equal the identity reflects our requirement that the m equivalent blocks of size $k + 1$ be created by inserting bars from left to right. The permutation $\sigma_i \in S_m$ for $i > 0$ specifies the order in which bars are inserted into the i-th slot in the m blocks. By i-th slot, we mean the bar position separating i objects to its left from $k + 1 - i$ objects to its right. For example, the facet in Figure 4 has permutations $12, 12, 21, 21, 12$ and the facet of Figure 5 is given by permutations $123, 123$ and 321.

Definition 5.2. *We refer to the k positions into which bars may be inserted in the m blocks as the* **slots***, and we call the ranks at which one of the m blocks is refined* **slot ranks***. Let us refer to the $m - 1$ bars inserted left to right initially to separate the m equivalent blocks as the* **splitters***. We say that a face includes a splitter if it includes the rank at which that splitter is inserted.*

The facet given by $k + 1$ copies of the identity permutation will have the empty set as its partitioning minimal face. For other facets, we are interested in finding minimal rank sets needed to differentiate them from this "identity facet".

Remark 5.1. *The effect of including a splitter in a face is to distinguish blocks to its left from blocks to its right. Including slot ranks from two different slots may record the fact that two blocks are split in opposite order in the two slots. For example, ranks 8,16 in Figure 7 show that blocks 1 and 2 are filled in opposite orders in the first and third slots.*

Figure 5 gives an example of a facet which does not have the same face of support $1,8$ as the identity facet. Now let us specify how to assign minimal faces G_i to the facets F_i, using the representation of facets as $(k+1)$-tuples of permutations $\sigma_0, \ldots, \sigma_k \in S_m$.

Partitioning Construction 5.1. First we construct permutations $\pi_0, \ldots, \pi_k$ from a facet F_i. Let π_{i+1} be the permutation in two-line notation which has σ_i (written in one-line notation) as the first line and σ_{i+1} (again in one-line notation) as the second line. The "wrap-around" permutation π_0 is obtained by using σ_k as the first line and σ_0 as the second line. The minimal face G_i associated to F_i consists of the ranks $lm + j$ such that $\pi_l^{-1}(j) > \pi_l^{-1}(j+1)$.

Example 5.1. *Letting $k = 2, m = 10$, Figure 6 depicts the facet F_i given by permutations $\sigma_0 = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, $\sigma_1 = 1, 2, 3, 7, 8, 9, 10, 4, 5, 6$ and $\sigma_2 = 1, 2, 3, 9, 10, 4, 5, 6, 7, 8$. The minimal face assigned to F_i has support $8, 17, 28$.*
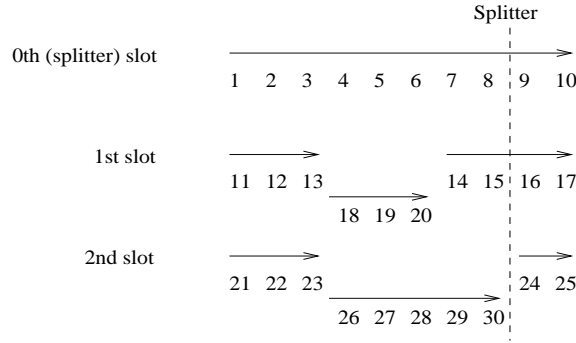

FIGURE 6. Facet with minimal face of support $8, 17, 28$

To ensure that our assignment of minimal faces to facets gives a partitioning, we must check (1) that every face belongs to some interval $[G_i, F_i]$ and (2) that no face is included in two different intervals. To verify (1), we describe how to extend any face F to a facet F_i whose minimal face G_i is contained in F.

Proposition 5.1. *Every face F is contained in at least one interval $[G_i, F_i]$.*

PROOF. First let us choose a representation for F, i.e. a specification of which blocks (among equivalent choices) to split to what extent at each slot rank in F. Figure 7 depicts our choice for a face of support $2, 3, 8, 16, 19$. By convention, we split blocks in a single slot as follows. First, we split equivalent blocks from left to right in the last slot. At this first step, two blocks are equivalent if there are no splitters separating them. Now split each set of equivalent blocks in the penultimate slot from left to right. At this second step, two blocks must also have been split at the same rank in the last slot in order to be equivalent. Proceed in this fashion through all the slots in reverse order to obtain a representation of F. Thus, F is encoded as a choice of which actual blocks are split at each slot rank of F, along with the list of which splitter ranks belong to F.

Now F is extended to a facet $\overline{F}$ by refining the order in which bars are inserted as follows: extend each interval between ranks in the first slot by splitting blocks from left to right. This choice determines the permutation σ_1. Similarly refine slot $s + 1$, but instead of extending each interval from left to right, proceed in the order that is increasing with respect to the permutation σ_s. This ensures that π_s^{-1} will not have any descents on the intervals between consecutive slot ranks. We may, however, have some descents in π_0^{-1} at ranks that are not splitter ranks.

These ranks may be eliminated by modifying our face representation as follows. Whenever π_0^{-1} has a descent at a rank that is not a splitter rank, this means there must be a slot rank at which two consecutive blocks separated by this splitter cease to be equivalent. We reverse the roles of these two blocks in our modified face representation. The effect is to change the descent in π_0^{-1} to an ascent, and instead to have a descent at the slot rank where the blocks cease to be equivalent. Now taking the increasing extension of this new face representation yields a facet F_i such that the support of G_i is contained in the support of F, implying $F \in [G_i, F_i]$. A very similar argument, with more detail included, is used to partition $\Delta(B_{lm})/S_l \wr S_m$ in [He]. $\square$

Example 5.2. *Consider the link subcomplex Δ specified by the face listed at the top in Figure 7. Below this is our representation for a face F of support $2, 3, 8, 16, 19$ in Δ and then its extension to a facet F_i. We list the ranks at which bars are inserted, using slight variation in height to distinguish different slots. Observe from the descents in $\pi_0^{-1}, \pi_1^{-1}, \pi_2^{-1}$ that the minimal face G_i has support $3, 8, 16, 19$.*
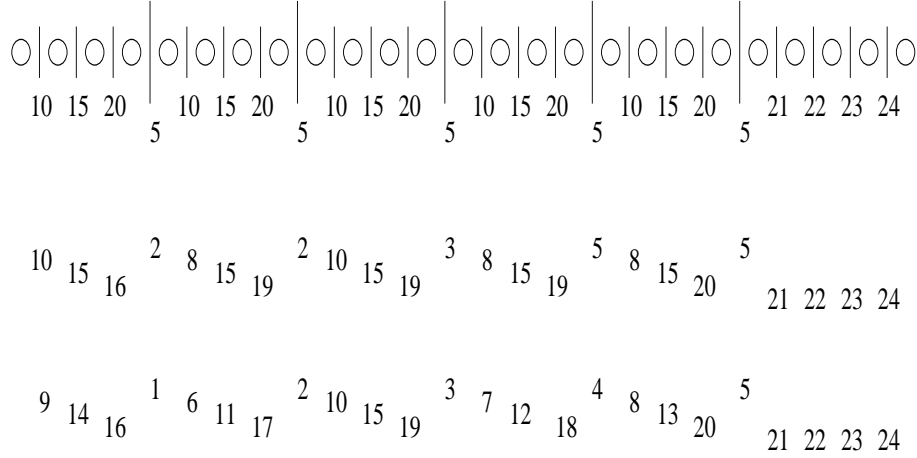

FIGURE 7. Face representation and extension

Next we check that each face is included only once in the partitioning.

Proposition 5.2. *There is no overlap among the intervals $[G_i, F_i]$.*

PROOF. Each representation of a face F has a unique extension to a facet that avoids descents between slot ranks in F. Only one of these representations will also avoid descents from wrap-around at ranks that are not splitter ranks, making our choice unique. Thus, there is only one extension of F to a facet F_j such that the support of F contains the support of G_j, as needed. $\square$

Question 5.1. *Is the incidence matrix for this matching of minimal faces with facets containing them nonsingular? If more generally this holds for a partitioning of $\Delta(\Pi_n)/S_n$, then the partitioning will give a ring invariant basic set for $k[\Delta(\Pi_n)]^{S_n}$, by results of [GS].*

5.2. **Partitioning subcomplexes of larger links.** Next we generalize the partitioning of the previous section to allow for examples such as the following:

Example 5.3. *Let $F_j, F_{i'}$ be the top and bottom facet in Figure 8, respectively. $F_j \cap (\cup_{i<j} F_i)$ has a maximal face G of support $2, 6, 7, 8, 9, 10, 11, 12$ since $G \in F_{i'}$ Consequently, F_j would need to contribute minimal faces of support $\{2, 3, 7\}, \{2, 4, 7\}$ and $\{2, 5, 7\}$ in a lexicographic shelling. This is resolved in a partitioning by assigning the faces of support $\{2, 3, 7\}$ and $\{2, 5, 7\}$ to lexicographically later facets.*
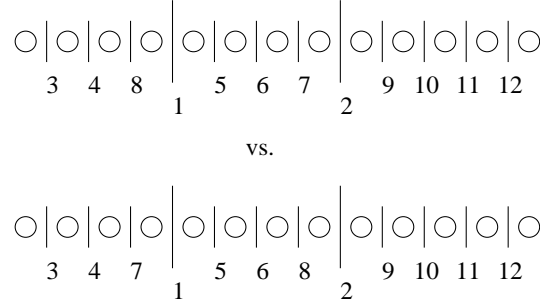

vs.

FIGURE 8. Two facets intersecting in faces of codimension one and three

Consider a face consisting of the following chain $u_1 < u_2 \prec u_3 \prec \cdots \prec u_r$ of partitions. Let u_1 consist of m equivalent blocks of size $k + 1$ and one block of size $l > k + 1$. Let u_2 have these m blocks refined to singletons and leave the block of size l unrefined. Finally, take a saturated chain $u_2 \prec u_3 \prec \cdots \prec u_r$ which at each stage splits off a singleton from its unique nontrivial block. For example, take the face of support $2, 8, 9, 10, 11, 12$ in either facet of Figure 8. Now we will consider the subcomplex of the link of such a face in which we require that splitters be inserted strictly from left to right. Denote this subcomplex by Δ.

Each facet in Δ inserts splitters left to right, then refines the blocks $B_1, \ldots, B_m$ entirely. Our assignment of minimal faces G_j to facets F_j will generalize the partitioning of Section 5.1, but it will use a single evolving permutation to play the roles of the permutations $\sigma_0, \ldots, \sigma_k$ and $\pi_0, \ldots, \pi_k$. Denote by σ this new permutation which keeps track of the order of our m blocks as they are progressively refined. We initialize σ to the identity (so blocks are initially ordered left to right). Each time a block B is refined, B is shifted to a later position in the evolving block order σ. Specifically, B is shifted to the last position among blocks which are similar to B in the sense of the following interwoven definitions.

Definition 5.3. A series of consecutive covering relations $u_0 \prec \cdots \prec u_{km}$ is called a **similarity series** if there is some collection of blocks $B_1, \ldots, B_m$ that are similar at u_0 and have each been split in the same fashion in the saturated chain from $\hat{0}$ to u_0. We also require for each $0 \le i < m$ that the covering relations $u_{ik} \prec \cdots \prec u_{(i+1)k}$ split the block B_{i+1} in a fashion that avoids topological descents (and also avoid ranks that would be included in the minimal face for a partitioning restricted to B_{i+1}). Furthermore, the blocks must be split in identical fashion within a similarity series.

The requirement about avoiding ranks that would be included in a partitioning minimal face is discussed more just prior to Theorem 5.1. By definition, similarity series' are non-overlapping.

Definition 5.4. Let us define **similarity** of blocks recursively as follows. When m consecutive ranks insert bars from left to right in a single block creating m left children of equal size, these children are at this point all similar. A collection of blocks $B_1, \ldots, B_t$ which are similar at u will still be similar at v for $u < v$ if every time any one of the blocks B_i appears in the interval from u to v, it appears as part of a similarity series for $B_1, \ldots, B_t$ (though this similarity series might continue beyond v or begin prior to u).

Notice that a block $B_i \in \{B_1, \ldots, B_m\}$ ceases to be similar to the other blocks when two bars are inserted in consecutive steps in B_i as a topological descent. Similarity is also broken at ranks which for some other reason are included in the minimal face assigned to the facet restricted to B_i in the partitioning for the complex restricted to B_i.

As an example, each facet in Figure 8 has two similar blocks at rank 2 which remain similar at rank 6. This similarity would have been broken if bars were instead inserted from right to left at ranks $3, 4$ or $5, 6$. Notice that when two similar blocks are split in identical fashion, but there are intermediate steps splitting other blocks in other ways, then block similarity is broken. Once B_i and B_j cease to be similar, σ henceforth preserves the relative order of B_i and B_j.

At each refinement step, σ orders blocks within each similarity class according to the order in which they have most recently been refined. The descents in σ_{final}^{-1} determine which splitters to include in the minimal face associated to a facet. At any particular rank, σ reflects the partial evolution based on block refinement up to this point. Now let us define what it means for two consecutive refinements $u \prec v, v \prec w$ to be increasing in the relative transpose order.

Definition 5.5. *Let us represent each of the km positions into which bars may be inserted by a pair (B, s) consisting of the block B being split and the slot s within B. The* **relative transpose order** *on bar positions (B, s) satisfies $(B, s) < (B', s')$ if (1) $s < s'$ or (2) $s = s'$ and $\pi(B) < \pi(B')$, evaluating π at u, i.e. just prior to both of the labels to be compared.*

Descents in the relative transpose order indicate which ranks to include in assigning minimal faces to facets in an analogous fashion to Section 5.1. As an example, the first facet in Figure 8 has a descent at rank 4 in the relative transpose order, but not at ranks 3 or 5, leading us to assign the minimal face of support $1, 4, 7$ to the first facet of Figure 8.

When two bars are inserted in consecutive steps into a block $B_i \in \{B_1, \ldots, B_m\}$, we turn to a partitioning for the quotient complex given by restriction to B_i to decide whether to include the rank in the minimal face assigned to our facet. In particular, the rank is included as topological descent unless the restriction gives a non-shelling step in the lexicographic order on $\Delta(\Pi_{|B_i|})/S_{|B_i|}$, in which case we turn to embedded instances of partitioning link subcomplexes.

Theorem 5.1. *The link subcomplex Δ defined in this section is partitionable.*

PROOF. The minimal face G_j assigned to the facet F_j is obtained by restricting to the following ranks of F_j.

- any slot rank separating consecutive bar insertions into distinct blocks $B_i, B_j \in \{B_1, \ldots, B_m\}$ which is a descent in the relative transpose order on bar positions.

- any slot rank separating consecutive bar insertions into a single block $B_i \in \{B_1, \ldots, B_m\}$ which would be included in the minimal face assigned to the facet restricted to $\Delta(\Pi_{|B_i|})/S_{|B_i|}$ in the partitioning of the quotient complex $\Delta(\Pi_{|B_i|})/S_{|B_i|}$.
- any splitter such that σ_{final}^{-1} has a descent at the splitter location.

With these choices, the arguments of Section 5.1 generalize. Namely, each face F extends by increasing chains in the relative transpose order to a unique facet F_i such that $F \in [G_i, F_i]$. Begin by splitting the leftmost of equivalent blocks at each rank. Then extend to a facet $\overline{F}$ that is increasing in the relative transpose order on each interval, except perhaps for descents from wrap-around. Next, permute the blocks in between consecutive splitters included in F so that σ_{final} is increasing in between splitters. Now extend each interval in the unique way that is increasing in the relative tranpose order to obtained the desired F_i with the property that $F \in [G_i, F_i]$. □

More generally, we will also need to account for non-shelling steps in which a set of similar blocks is partially refined, then other blocks are refined before further refining the set of equivalent blocks. Figure 9 gives an example of a face from such a non-shelling step. Partitioning must accomodate any number of such alternations.
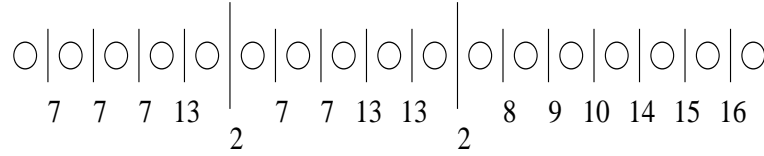

$$7 \quad 7 \quad 7 \quad 13 \qquad 7 \quad 7 \quad 13 \quad 13 \qquad 8 \quad 9 \quad 10 \quad 14 \quad 15 \quad 16$$
$$2 \qquad\qquad\qquad 2$$

FIGURE 9. A face with nonconsecutive steps splitting equivalent blocks

Remark 5.2. Partitioning link subcomplexes in which the similar blocks need not be refined entirely in consecutive steps is done as in Theorem 5.1. That is, whenever, two steps refining similar blocks are separated by a step refining an outside block B, there is a topological descent either in departing from the collection of similar blocks or in returning to it, because the similar blocks occur consecutively (as will be verified in Lemma 5.1). This descent breaks the similarity of two blocks when one is refined before the refinement of B and the other is refined afterwards. This is the only amendment to the argument of Theorem 5.1 which is needed.

5.3. **Extension to a partitioning for** $\Delta(\Pi_n)/S_n$. Let us now characterize all non-shelling steps and show how to merge the partitionings of Section 5.2 into a partitioning for all of $\Delta(\Pi_n)/S_n$. By Theorem 4.1, a non-shelling step may only arise when skipping some minimal rank set $i_1, \ldots, i_k$ renders two blocks B, B' in a facet F_j equivalent, and the block B or B' which is farther to the right is split first by F_j. By the increasing chain condition (cf. Theorem 4.1), the blocks B, B' must be created within a sequence of consecutive refinement steps that create a set of identical blocks from left to right from a single parent.

Lemma 5.1. *Similar blocks are positioned consecutively.*

PROOF. Consecutiveness follows from the fact that the blocks must be created from a single parent in consecutive steps with no topological descents; the one case requiring special care is when the rightmost of similar blocks is larger than the

other block created in the same step, so then a smaller block comes in between $m - 1$ of the similar blocks and the last one, but this gives a topological descent immediately before the bar splitting off the last of the identical blocks, rendering this last block not equivalent to the others.   □

When the last two blocks created (in a single step) have equal size, then these two blocks are not just similar, but actually equivalent. This implies that the bar between them cannot be a splitter, in that the inclusion of this rank does not distinguish the two ranks. By convention, we always must refine the left of these two equivalent blocks first, until their equivalence is broken by the inclusion of a slot rank. When the last slot does not have this rightmost block filled last, then this gives a descent in the inverse to the wraparound permutation. By convention, we then include in the associated minimal face the rank in the first slot (rather than the forbidden rightmost splitter in the 0-slot).

Because the set of m similar blocks appear consecutively, any two facets sharing a link of the type considered in Section 5.2 will have the same topological descents among the ranks outside the link. This ensures that all faces in a particular link subcomplex will include the same external ranks in their minimal faces both from descents and also from other external partitionings. In conclusion, we get a partitioning for $\Delta(\Pi_n)/S_n$, as confirmed next.

Theorem 5.2. *The quotient complex $\Delta(\Pi_n)/S_n$ is partitionable.*

PROOF. For each facet F_j which is inserted as a lexicographic shelling step as in Section 4, let G_j be the face comprised of the topological descents in F_j. By Theorem 4.2, every nonshelling step F_j has at least one series of consecutive bar insertions creating similar blocks by inserting splitters from left to right. For each such collection of similar blocks, use the partitioning of the link subcomplex with the appropriate m, k and with the partial refinements that reflect alternation between refining the m blocks and the blocks outside the segment (cf. Remark 5.2) to determine which ranks to include in G_j among the ranks belonging to the link.

Among the ranks of F_j not belonging to any such link subcomplex, include in G_j the topological descents according to the lexicographic order on facets given in Section 4. To be sure this this assignment of minimal faces to facets gives a partitioning, we need the fact that facets involved in the same link the same topological descents outside of the link. This follows from the characterization of non-shelling steps, and in particular from the fact that equivalent blocks must occur consecutively. Thus, any fixed bar insertion outside the collection of equivalent blocks will either be to the right of all possible bar positions within the collection of equivalent blocks or to the left of them all. Thus, the non-shelling steps collectively contribute exactly the faces needed to complement the lexicographic shelling steps. This follows because for any non-shelling step, all the minimal faces that first appear lexicographically in it are obtained by taking a minimal face in the partitioning of a link subcomplex, as in Section 5.2, and adding the ranks outside the link which are topological descents or are chosen in another link subcomplex partitioning. □

6. Obtaining the multiplicity of the trivial representation from a quotient complex shelling or partitioning

Following [St2], denote by $\alpha_S(\Pi_n)$ the symmetric group action on chains with rank set S. Let $\beta_S(\Pi_n)$ be the symmetric group representation on top homology

of the rank-selected partition lattice with ranks belonging to S. Information about the flag f-vector and flag h-vector may be found (for instance) in [St4]. Recall,

$$\beta_S = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T$$

where β_S is the induced representation on rank-restricted homology, since Π_n is Cohen-Macaulay. Note that the multiplicity $\langle \alpha_T, 1 \rangle$ of the trivial representation in $\alpha_T(\Pi_n)$ is the number of orbits in the induced representation on chains with rank set T, namely $\langle \alpha_T, 1 \rangle = f_T(\Delta(\Pi_n)/S_n)$. This together with inner product linearity yields

$$\begin{aligned}
\langle \beta_S, 1 \rangle &= \sum_{T \subseteq S} (-1)^{|S-T|} \langle \alpha_T, 1 \rangle \\
&= \sum_{T \subseteq S} (-1)^{|S-T|} f_S(\Delta(\Pi_n)/S_n) \\
&= h_S(\Delta(\Pi_n)/S_n)
\end{aligned}$$

where h_S is the term indexed by S in the flag h-vector for $\Delta(\Pi_n)/S_n$. Recall that $h_S(\Delta)$ counts the minimal new faces colored by S in any shelling (or partitioning) for a balanced boolean cell complex. Thus, a shelling yields an interpretation for the multiplicity $\langle \beta_S, 1 \rangle$ of the trivial representation in $\beta_S(\Pi_n)$. Furthermore, the collection of colors in the minimal new face in a lexicographic shelling step is the collection of ranks where (topological) descents occur in the facet being inserted.

The above discussion applies to any rank-preserving group action on any finite, ranked Cohen-Macaulay poset with a shelling or partitioning. Thus, we also get an interpretation for the multiplicity of the trivial representation in the $S_2 \wr S_n$ action on the homology of the rank-selected boolean lattice B_{2n} from our shelling for $\Delta(B_{2n})/S_2 \wr S_n$.

7. Predicting when $b_S(n) = 0$

The partitioning for $\Delta(\Pi_n)/S_n$ yields a combinatorial interpretation for b_S as the number of facets with minimal new face colored by S in the partitioning for $\Delta(\Pi_n)/S_n$. The following appeared as Conjecture 4.11 in [BK].

Conjecture 7.1 (Babson-Kozlov). *The rank-selected quotient complex $\Delta(\Pi_n^S)/S_n$ is contractible if and only if $b_S(n) = 0$.*

Notice that $b_S(n) > 0$ if and only if the partitioning for $\Delta(\Pi_n)/S_n$ has a minimal face of support S. A shelling for $\Delta(\Pi_n)/S_n$ would resolve the conjecture affirmatively.

Next, we recover a result of Hanlon [Ha] (see [Su], [Ko1] for proofs by other methods). Our argument is included to give a concrete example of how the partitioning for $\Delta(\Pi_n)/S_n$ may lead to results about $\langle \beta_S, 1 \rangle$. More extensive results of this nature, including the confirmation of two conjecture from [Su], have recently been developed in a joint project with Phil Hanlon [HH].

Our proof below resembles the one given by Kozlov in [Ko1] in the sense that we both show that $h_{1,\ldots,i}(\Delta(\Pi_n)/S_n) = 0$ and use the fact that $h_S(\Delta(\Pi_n)/S_n) = \langle \beta_S, 1 \rangle$. However, Kozlov's proof is topological and ours is combinatorial. More specifically, Kozlov uses the interpretation of $h_S(\Delta)$ as the reduced Euler characteristic for Δ^S, deducing that it is 0 from the fact that $\Delta(\Pi_n^{1,\ldots,i})/S_n$ is collapsible.

We instead use the fact that $h_S(\Delta)$ counts boolean intervals with minimal element of support S in a partitioning for Δ.

To show $b_S(n) = 0$ for a particular S, we will show that S never occurs as a descent set in a shelling step or as the support of the minimal new face assigned to some non-shelling step. We work in terms of the dual poset to the partition lattice as considered in [Ha] and [St2]. To account for this, we must replace each $i \in S$ by $n - i - 1$. Thus, the next theorem requires that we show there are no shelling steps that begin with entirely (topological) ascents and then consist entirely of (topological) descents as soon as the first (topological) descent occurs; we must also check a similar condition for the non-shelling steps.

Theorem 7.1. *The multiplicity $b_S(n)$ of the trivial representation in β_S is 0 for $S = \{1, \ldots, i\}$ and $1 \le i \le n - 2$.*

PROOF. First consider the facets that are inserted as shelling steps. Recall our convention of placing bars as far to the left as possible at each step. A facet with descent set $n - i - 1, n - i, \ldots, n - 3, n - 2$ could only come from initially inserting bars left to right creating blocks nondecreasing in size; after this, we would need to completely refine the rightmost block using only topological descents and then proceed similarly through the remaining blocks from right to left. If the initial insertion of bars left to right created any blocks of size greater than 2, it would be impossible to later refine such a block with only topological descents. Thus, we may assume bars are initially placed left to right creating blocks entirely of size 1 followed by blocks entirely of size 2, as in Figure 10. However, notice that the two

FIGURE 10. The impossibility of descent set $\{n - i - 1, n - i, \ldots, n - 1\}$

rightmost blocks of size 2 will be equivalent to each other, which means the left one must be split first. This necessitates the existence of a (topological) ascent at some point after the first (topological) descent, as in Figure 10, so we are done with the shelling steps.

The non-shelling steps all have support including ranks other than only the final string by virtue of creating equivalent blocks to be partitioned from left to right. A descent is necessitated by the insertion of a bar creating the rightmost of the equivalent blocks farther to the right than bars to be inserted within the equivalent blocks; there will be an ascent at some point after this descent since the partitionings of link subcomplexes cannot have facets with minimal faces of full support. □

8. AN APPLICATION TO SUB-RINGS OF INVARIANT POLYNOMIALS

Let us review notation from [GS] needed to state a theorem of Garsia and Stanton (which appears as Theorem 6.2 in [GS] and is restated as Theorem 8.1 below). Let C be a balanced boolean cell complex consisting of vertices $x_1, \cdots, x_n$, ordered in a way that is compatible with their colors; that is, we choose a vertex order such

that x_i is colored with a smaller number than x_j for all $1 \leq i < j \leq n$. If c is a face of C consisting of vertices $x_{i_1}, \cdots, x_{i_k}$, then denote by $x(c)$ the monomial $x_{i_1} \cdots x_{i_k}$. When a group H acts on C, let

$$R^H(x(c)) = \frac{1}{|H|} \sum_{h \in H} h x(c) = \frac{1}{|H|} \sum_{h \in H} x(hc).$$

Let $\theta_i = \sum_{c(v)=i} x(v)$, a sum over vertices in C of color i. A set of chain monomials $\{x(b) | b \in B\}$ given by a collection B of chains in a poset P is called a basic set if every element Q of the Stanley-Reisner ring R_P has a unique expression

$$Q = \sum_{b \in B} x(b) Q_b(\theta_1, \ldots, \theta_d)$$

where the coefficients $Q_b(\theta_1, \ldots, \theta_d)$ are polynomials with rational coefficients in the variables $\theta_1, \cdots, \theta_d$. All Cohen-Macaulay posets have such basic sets, and [GS] shows how a quotient complex shelling gives an explicit basic set.

Theorem 8.1 (Garsia and Stanton). *If C/H has a shelling $F_1, \ldots, F_k$ where $R(F_i)$ is the minimal new face in F_i and b_i is a representative within C of the orbit of $R(F_i)$, then the orbit polynomials $R^H x(b_i)$ form a basic set for $R^H R_P$.*

Thus, our lexicographic shelling for $\Delta(B_{2n})/S_2 \wr S_n$ yields a ring invariant basic set. A simple description of which descent sets occur in the lexicographic shelling would be desirable since it would yield a nice description of these ring invariant basic sets, according to Theorem 8.1. It would also be interesting to determine whether the incidence matrix given by our partitioning for $\Delta(\Pi_n)/S_n$ is nonsingular, since that would imply a basic set for $k[\Delta(\Pi_n)]^{S_n}$, by another result of [GS].

Stanley showed in [St1] that the face ring of a Cohen-Macaulay simplicial poset is a Cohen-Macaulay ring. According to Proposition 1.2, our shelling for $\Delta(B_{2n})/S_2 \wr S_n$ implies that the face ring $k[\Delta(B_{2n})/S_2 \wr S_n]$ is Cohen-Macaulay (over the integers). However, Reiner constructed an isomorphism in [Re] between the face ring $k[\Delta(P)/G]$ and the ring of invariants $k[\Delta(P)]^G$. Reiner's result allows us to conclude that the invariant subring $k[B_{2n}]^{S_2 \wr S_n}$ is Cohen-Macaulay over the integers.

Furthermore, Reiner showed that in the case of quotients of the Boolean algebra by a permutation subgroup G of S_n, the invariant subring of the polynomial ring $k[x_1, \ldots, x_n]$ under the action of G is Cohen-Macaulay over R whenever the same is true for the invariant subring $R[B_n]^G$ of the Boolean algebra's face ring over R; Reiner recently provided a proof for this (formerly unpublished) result as an appendix to [He]. Since $S_2 \wr S_n$ is such a permutation subgroup of S_{2n}, we may conclude the following from our shelling for $\Delta(B_{2n})/S_2 \wr S_n$.

Theorem 8.2. *The subring of invariant polynomials $k[x_1, \ldots, x_{2n}]^{S_2 \wr S_n}$ is Cohen-Macaulay for any field k.*

When k is the field of complex numbers, this is a special case of a result from [HE], but the shelling gives the Cohen-Macaulay property also for fields of finite characteristic, or equivalently for integer coefficients.

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